British Astronomy and Astrophysics Olympiad 2023-2024

Astro Round 1 Section 2

Tuesday 23rd January 2024

This question paper must not be photographed or taken out of the exam room

Instructions

Time: 1 hour.

Questions: Answer any **TWO** questions, worth 20 marks each to give a total of 40 marks for this section (there are FIVE questions available).

Solutions: Answers and calculations are to be written on loose paper. START EACH QUESTION ON A NEW PAGE. Students should ensure their **name** and **school** is clearly written on the **first** answer sheet and that **all** pages are numbered. A standard formula booklet with standard physical constants may be used if desired.

Clarity: Solutions must be written legibly, in black pen, and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam.

Calculators: Any standard calculator may be used, but calculators cannot be programmable and must not have symbolic algebra capability.

Sitting the paper: There are two options for sitting the Astro Round 1,

1. Section 1 and Section 2 may be sat in one session of 2 hours. Section 1 should be collected in after 1 hour and only then should Section 2 be given out.
2. Section 1 and Section 2 are sat in entirely separate sessions of 1 hour each.

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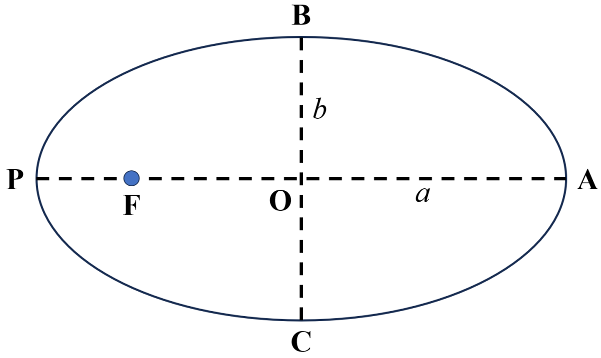
Important Constants

Constant	Symbol	Value
Speed of light	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16}\,\mathrm{m}$
Astronomical Unit	au	$1.50 \times 10^{11}\,\mathrm{m}$
Radius of the Sun	$R_{\odot}$	$6.96 \times 10^{8}\,\mathrm{m}$
Radius of the Earth	$R_{\oplus}$	$6.37 \times 10^{6}\,\mathrm{m}$
Mass of the Sun	$M_{\odot}$	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	$M_{\oplus}$	$5.97 \times 10^{24}\,\mathrm{kg}$
Luminosity of the Sun	$L_{\odot}$	$3.83 \times 10^{26}\,\mathrm{W}$
Absolute magnitude of the Sun	$\mathcal{M}_{\odot}$	4.74
Hubble constant	H_0	70 km s^{-1} Mpc^{-1}
Stephan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Gravitational constant	G	6.67×10^{-11} m^{3} kg^{-1} s^{-2}
Boltzmann constant	k_{B}	1.38×10^{-23} J K^{-1}
Permittivity of free space	ε_0	8.85×10^{-12} F m^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Planck's constant	h	6.63×10^{-34} J s
Elementary charge	e	1.60×10^{-19} C
Proton rest mass	m_{p}	1.67×10^{-27} kg
Electron rest mass	m_{e}	9.11×10^{-31} kg
Wien's displacement law	$\lambda_{\mathrm{max}}T$	2.90×10^{-3} m K
Avagadro's constant	N_A	6.02×10^{23} mol^{-1}

Important Formulae

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit:

$a = \mathbf{OA}\ (= \mathbf{OP})$	semi-major axis
$b = \mathbf{OB}\ (= \mathbf{OC})$	semi-minor axis
$e = \sqrt{1 - \frac{b^2}{a^2}}$	eccentricity
$\mathbf{F}$	focus
$\mathbf{PF} = a(1-e)$	periapsis distance (shortest distance from $\mathbf{F}$)
$\mathbf{AF} = a(1+e)$	apoapsis distance (longest distance from $\mathbf{F}$)
πab	area of the ellipse

Kepler's Third Law:

$$T^2 = \frac{4\pi^2}{GM} a^3$$

Vis-Viva Equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Wien's Displacement Law:

$$\lambda_{\max} T = \text{constant}$$

Stephan-Boltzmann Law:

$$L = 4\pi R^2 \sigma T^4$$

Brightness (Intensity):

$$b = \frac{L}{4\pi d^2}$$

Magnitudes:

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)}$$

$$m - \mathcal{M} = 5 \log\left(\frac{d}{10}\right)$$

Distance-Parallax Relation:

$$d = \frac{1}{p}$$

Rayleigh Criterion:

$$\theta = \frac{1.22\lambda}{D}$$

Redshift:

$$z = \frac{\Delta\lambda}{\lambda_{\text{emit}}} \approx \frac{v}{c}$$

Hubble's Law:

$$v = H_0 d$$

Qu 2. Inside a Planet

In this paper there are FIVE questions and you should attempt only TWO.

The gravitational field strength, g, and gravitational potential, V_G, for distance $r \geq R$ where R is the radius of a planet is independent of the structure of a planet, as the planet can be considered a point mass, but for $r < R$ things can get interesting!

In this question we will consider the functional forms of the equations for g and V_G and their graphs for several different structures. Throughout, for simplicity, consider just the **magnitude** of g and V_G.

In the late 17$^{\text{th}}$ century, Edmund Halley (of comet fame) was one of several famous astronomers and physicists that considered the Earth to be hollow - suggesting that below a thin crust is a substantially empty space. Although thoroughly discounted within decades, it still finds an audience in fiction with ideas of subterranean lands and this is the first model we will explore.

a. Assume that the Hollow Earth theory is true and we live on a crust of negligible thickness compared to the radius of the Earth (and which contains all the mass of the Earth) and below it is a complete vacuum.

   (i) Derive algebraic expressions for g and V_G for $r < R_\oplus$. Justifications must be given using clear mathematical logic (a variety of approaches are possible).

   (ii) Hence sketch (on different axes) g and V_G for the Hollow Earth for $0 \leq r \leq 2\,R_\oplus$ with important **numerical** values indicated on the y-axis.

[4]

A Hollow Earth would need to be made of a material with far greater tensile strength than any we know of to not gravitationally collapse - most real structures are solid spheres. Our next models with look at this with different functions of density.

b. The simplest solid sphere model is one in which the Earth has a uniform density.

   (i) Derive algebraic expressions for g and V_G for $r < R_\oplus$.

   (ii) Hence sketch (on different axes) g and V_G for the Uniform Density Earth for $0 \leq r \leq 2\,R_\oplus$ with important **numerical** values indicated on the y-axis.

[8]

c. In reality, the core of the Earth is much denser than the crust, and so we can instead model the density as decreasing linearly with radius. For simplicity we will take it to be from a maximum at the centre, ρ_c, to zero at $r = R_\oplus$.

   (i) Derive an algebraic expression for g for $r < R_\oplus$.

   (ii) What is the maximum value of g and for what value of r (in units of $R_\oplus$) does it occur?

   (iii) Hence sketch g for $0 \leq r \leq 2\,R_\oplus$ with important **numerical** values indicated on the y-axis.

[8]

Qu 3. Jeans Mass

The Jeans mass is defined as the 'mass that a spherical cloud of interstellar gas must have to contract under its own weight', and is named after the British physicist Sir James Jeans. It is a fundamental part of the understanding of how stars form, but can be extended to any gravitational overdensity effect, such as that which led to the seeds of galaxy formation.

a. For a star of uniform density ρ with mass M and radius R, show that its total gravitational potential energy is $E_P = -\alpha \dfrac{GM^2}{R}$ where α is a rational constant to be determined.

[7]

The Virial Theorem states that for a stable system bound by a conservative force (such as the gravitational force), $E_K = -\frac{1}{2}E_P$ where E_K and E_P denote the total, time-averaged kinetic and potential energies of the system respectively.

b. Consider a nebula with uniform density ρ and of uniform temperature T. Derive an expression for the Jeans mass purely as a function of these two variables and various constants. Assume the nebula to be spherical and composed only of hydrogen atoms of mass m_H.
If you did not get a value for α in the previous part, take $\alpha = 1$.

[7]

c. The current estimate for the average (baryonic matter) density of the Universe is that it is equivalent to 1 hydrogen atom per 4 cubic metres. Consider a spherical region within the Universe of uniform density equal to this average and uniform temperature equal to that of the cosmic microwave background, and composed of only hydrogen atoms. Find the radius of this region such that it will collapse under its own weight in this very simple model, giving your answer in kpc. You are given the observed spectrum of the cosmic microwave background in Figure 1.

[6]

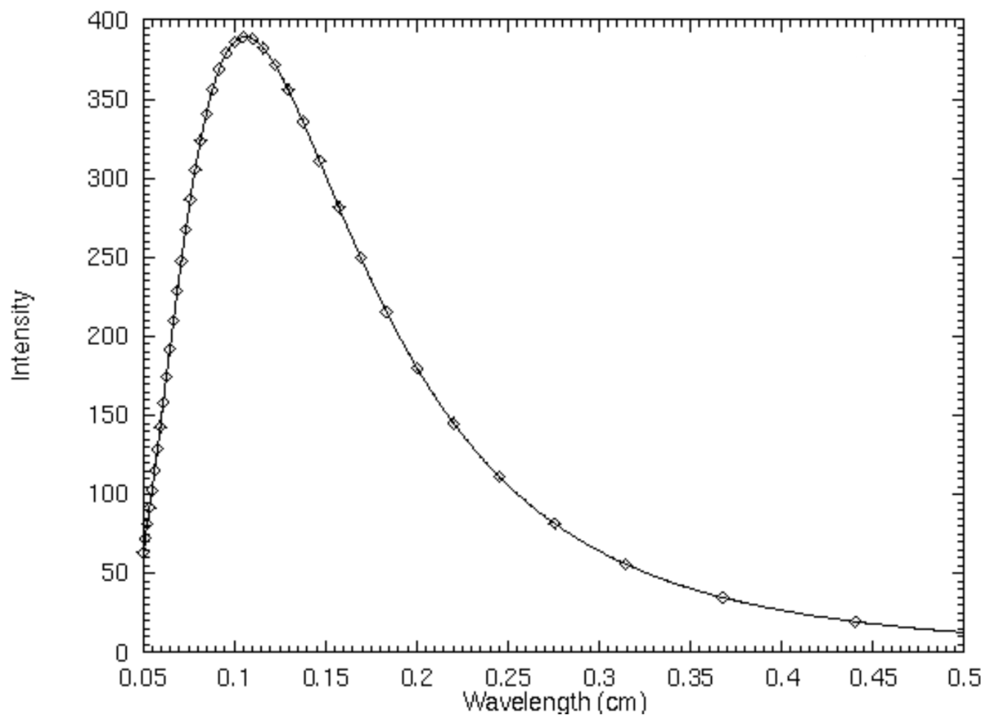


Figure 1: The observed spectrum of the cosmic microwave background.

Qu 4. Spherical Telescope

In this question you will examine the optics of Newtonian telescopes.

A Newtonian telescope uses a curved primary mirror to focus parallel incoming rays of light and a flat secondary mirror to divert the rays into the eyepiece. This is shown in Figure 2.

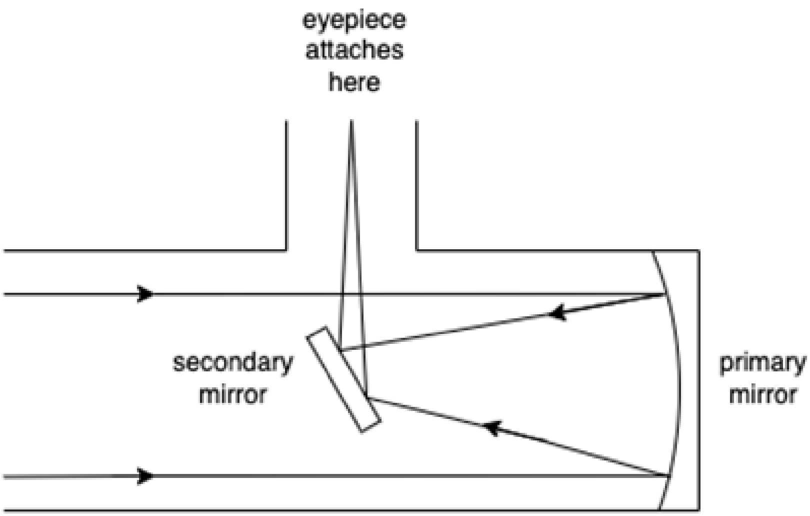


Figure 2: Newtonian telescope design.

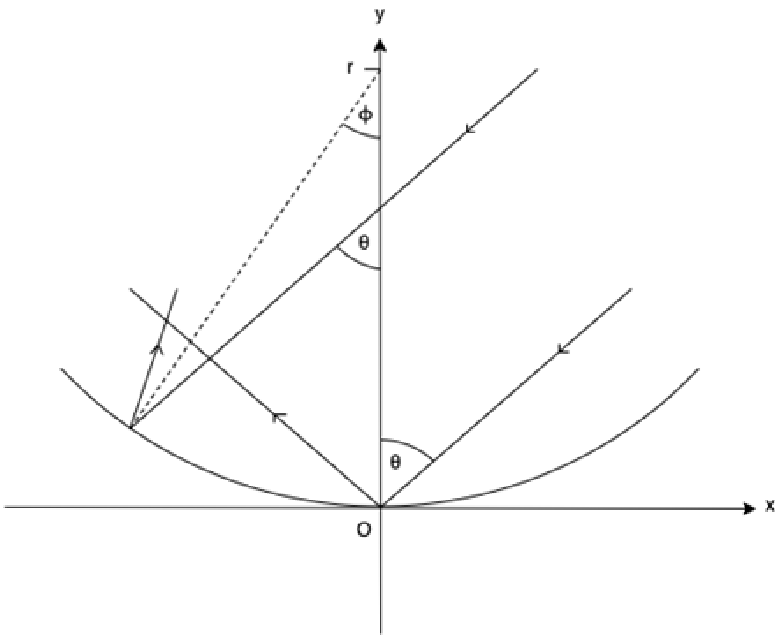


Figure 3: Ray diagram for a spherical mirror.

Consider a spherical primary mirror with the y-axis lying along the principal axis and the base of the mirror positioned at the origin, as shown in Figure 3. Rays of light are incoming from a point source very far away, such as a star, so that they are parallel.

a. If the incoming rays are incident at an angle θ to the principal axis, find an expression in terms of θ and the radius r of the mirror for the y-coordinate of the point where the reflected rays meet, forming an image of the point source. You may assume that r is much larger than the dimensions of the mirror. Use the diagram to help you - it may be useful to work out the Cartesian equations of the reflected rays.

You may use the tangent addition of angles identity, $\tan(\alpha+\beta) \equiv \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

[9]

b. Therefore, in the paraxial approximation (rays are incident at small angles to the principal axis), state the focal length of the spherical mirror.

[1]

A Newtonian telescope has a diameter $D = 20$ cm and its primary mirror has a focal length $f = 1.5$ m. The secondary mirror is inclined at $45°$ with respect to the principal axis, and eyepiece lenses can be attached to a tube that juts out $h = 5$ cm from the perimeter of the optical tube, in the same way shown in Figure 2.

c. How far from the primary mirror should the secondary mirror be placed to produce a focused image?

[4]

d. The telescope is used with an eyepiece of focal length 10 mm. Calculate the following:

(i) The f-number for this telescope.

(ii) The angular resolution of the telescope in arcseconds when observing at a wavelength of 600 nm (approximately that of visible light).

(iii) The factor by which the telescope increases light intensity.

[6]

Qu 5. Dark Energy

Dark energy is one of the most mysterious things in the Universe, seeming to dominate on the largest scales (see Figure 4) and responsible for the accelerating expansion of the Universe. Various models exist for what it might be, with the original (and still popular) model being that of a 'cosmological constant', Λ, which represents a vacuum energy density that is homogeneous throughout space and time. This vacuum energy is hypothesised to link to the production and annihilation of virtual particles in a 'vacuum' and so we can use Quantum Field Theory (QFT) to predict its size... however, the value from theory is very discrepant with what we observe from cosmology by a factor $\sim 10^{120}$! This question will get you to calculate the discrepancy for yourself.

In this question some numbers may go beyond your calculator's numerical limits for size, so you are encouraged to use logs if needed, and are required to do that in the final part of the question.

We shall begin by looking at the observed values from cosmology. The cosmological constant, Λ, as it appears in the Friedmann equations describing the evolution of the Universe, can be calculated as

$$\Lambda = 3\left(\frac{H_0}{c}\right)^2 \Omega_\Lambda \, ,$$

where H_0 is the Hubble constant (taken to be 70 km s^{-1} Mpc^{-1}), c is the speed of light, and

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_{crit}} \approx 0.7$$

with ρ_Λ equal to the (mass) density of the vacuum energy and ρ_{crit} being the critical (mass) density of the Universe for gravitational attraction to eventually overcome any initial expansion after the Big Bang.

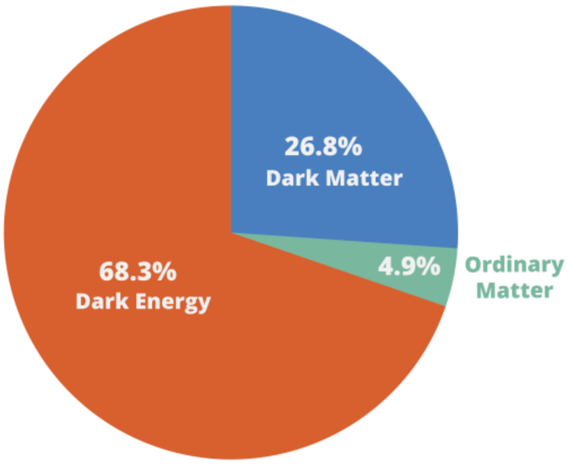


Figure 4: Percentage of ordinary matter, dark matter and dark energy in the universe, as measured by the final 2018 data release from the Planck satellite. Credit: E. Ward/ATLAS Collaboration.

a. Calculate a value for Λ today, giving your answer in SI base units. [Hint: you will need to convert H_0 into SI base units.]

[3]

b. Imagine a spherical section of a perfectly homogenous Universe expanding. If the density of the Universe is critical, all objects in it are moving at escape velocity. By considering the total energy of a test particle on the edge of the region with a speed given by Hubble's Law, show that the critical density $\rho_{crit} = \sigma \dfrac{H_0^2}{G}$ where σ is a constant that should be found in its exact form.

[4]

c. Evaluate ρ_{crit} for our Universe (note: this takes into account dark energy, dark matter and baryonic matter) in units of hydrogen atoms per cubic metre.

[2]

d. Hence, evaluate a value for ρ_Λ (given in units kg m^{-3}).

[1]

QFT has been fantastically successful at predicting and explaining results at the quantum level and so it was widely expected to be able to predict a value of the vacuum energy density, u_{vac}, which could be converted into a mass density and compared to the observed value.

The vacuum energy density can be calculated as

$$u_{vac} = \frac{\hbar}{8\pi^2 c^3}\omega_{max}^4 \qquad \text{where} \qquad \hbar \equiv \frac{h}{2\pi}$$

with h as Planck's constant and ω_{max} is the maximum angular frequency of vacuum fluctuations (based on quantum harmonic oscillators) and can be related to an energy of the fluctuation as $E = \hbar\omega_{max}$. In QFT, the largest value the energy of a single fluctuation can take is the Planck energy, E_{Pl}, so this will define the dominant component of u_{vac}.

e. Given $E_{Pl} = \hbar^\alpha c^\beta G^\gamma$, use dimensional analysis to determine α, β, and γ, and hence evaluate E_{Pl}, giving your answer in joules.

[4]

f. Use this to calculate ω_{max} and hence u_{vac}, stating your units clearly.

[4]

g. Convert u_{vac} to a mass density, ρ_{vac} (given in units kg m^{-3}).

[1]

h. Hence, show that $\log_{10}\left(\frac{\rho_{vac}}{\rho_\Lambda}\right) \sim 120$.

[1]

Qu 6. Observational Astronomy

Below is a map of the sky from Rio de Janeiro, Brazil on an unknown date. You may draw on it to help you answer the questions, although only what is on your loose paper will be marked.

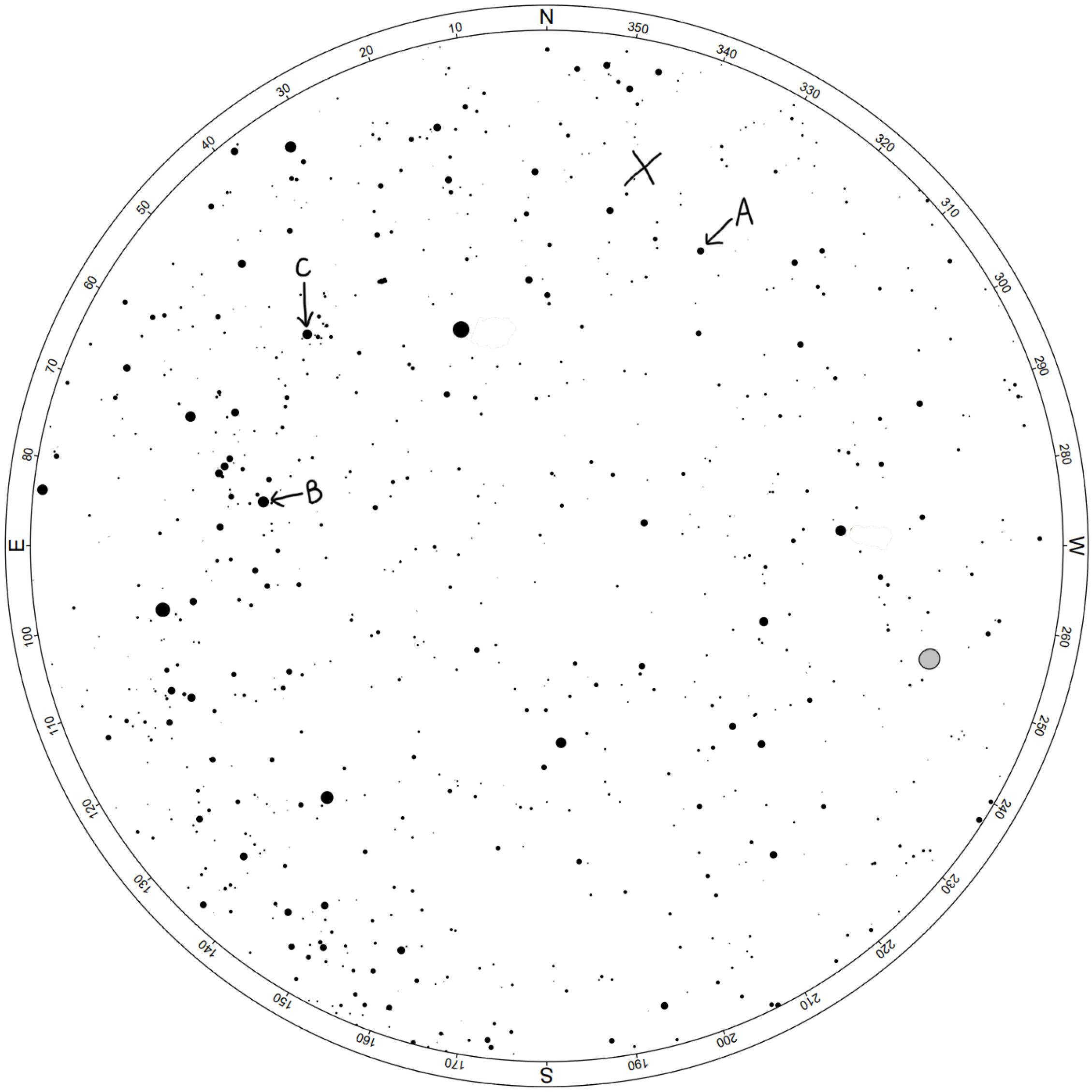


Figure 5: Sky map for use with Question 7. Altitude increases linearly from the horizon to the zenith in this figure. Stars to magnitude +4.7 are shown.

a. Name the star being pointed at with the arrows labelled A, B and C.

[3]

b. Estimate the altitude of Sirius to the nearest 2° and state the angle marker (from the scale around the circumference) that it is closest to.

[2]

c. Name three fully visible (i.e. fully above the horizon) **non-zodiacal** constellations that the galactic equator passes through in this image.

[3]

d. Jupiter and Saturn are both visible in this image. Name the two nearest zodiacal constellations to **Jupiter**.

[2]

e. The Moon is indicated as a grey circle in the West. Which zodiacal constellation is it in?

[1]

f. Consider the following list of Messier objects: M1, M31, M42, M45, M57, M76.

  (i) For **each** of them, state if they are ABOVE or BELOW the horizon in this image.

  (ii) Give the name and Messier number of the deep sky object whose location is indicated with a cross.

[5]

g. This is the sky at 03:19 at the location. Estimate the month. Credit will only be given for a well-explained answer.

[4]

END OF PAPER

Questions proposed by:
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