# Astro Round 1 2024 Paper Solutions

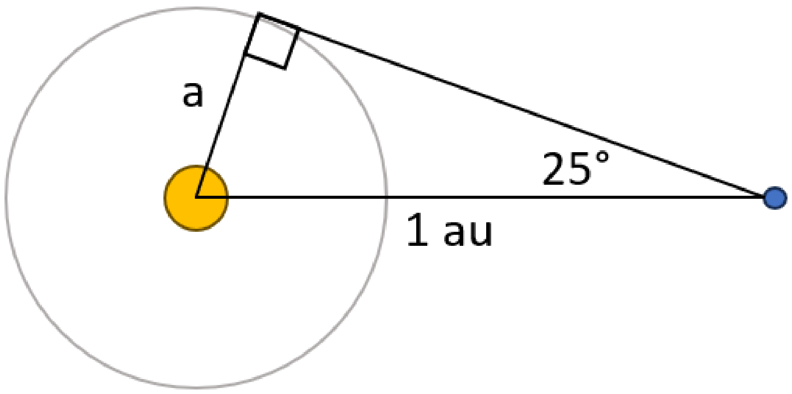
**Note for markers:**

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate. Note that sometimes the final mark can only be given if the answer is in the specified units, otherwise any equivalent units are fine
- Allow ecf with penalty only applied at each distinct mistake.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer
- Students getting the answer in a $\boxed{\text{box}}$ will get all the marks available for that calculation / part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made (students may not explicitly calculate the intermediate stages and should not be penalised for this so long as their argument is clear).

## Section 1 Mark Scheme – Q1 [Short Questions]

**A.**

Diagram showing both the 25° and 90° angle in the right place [1]

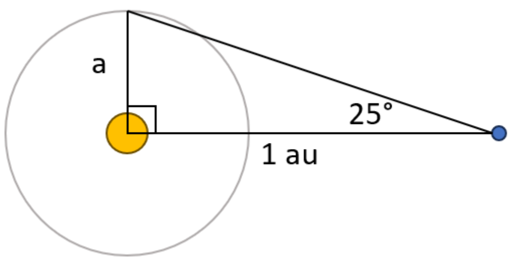



$\therefore a = 1\text{ au} \times \sin 25° = 0.423\text{ au}\ (= 6.34 \times 10^{10}\text{ m})$ [1]

$\therefore T = \sqrt{a^3} = \sqrt{0.423} = 0.275\text{ years} = \boxed{100\text{ days}}$ (must be in days for the mark) [1] [3]

[Allow ecf on 2nd marking point from their diagram, and 3rd mark is for correct application of Kepler's 3rd law for their value of a. Using the full SI version of Kepler 3 gets $\boxed{101\text{ days}}$ – this receives full marks. A common mistake is to draw the diagram as:

From this they can get an answer of 117 days (from a = tan 25°) which scores max 2 marks only.]



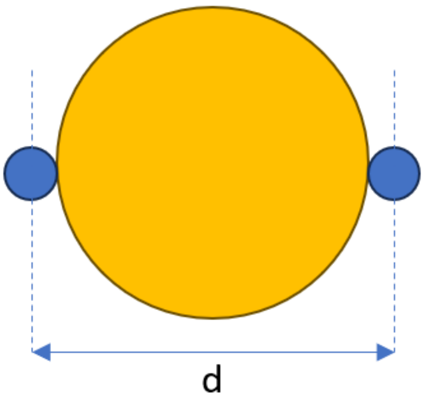
**B.**

The speed of the Earth in its orbit:

$$v = \sqrt{\frac{GM_{\odot}}{a_{\oplus}}} = \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{1.50\times10^{11}}} = 2.97\times10^{4}\ \mathrm{m\ s^{-1}} \qquad [1]$$

(accept $v = \frac{2\pi a_{\oplus}}{T} = \frac{2\pi\times1.50\times10^{11}}{365\times24\times60\times60} = 2.99\times10^{4}\ \mathrm{m\ s^{-1}}$)

The distance the centre of the Earth has to travel between start and end of the transit:




$$d = 2R_{\odot} + 2R_{\oplus} = 1.40\times10^{9}\ \mathrm{m} \qquad [1]$$

(Since we did not specify the precise start of the transit, accept $d = 2R_{\odot} = 1.39\times10^{8}$ m as well)

$$t = \frac{d}{v} = \frac{1.40\times10^{9}}{2.97\times10^{4}} = 4.72\times10^{4}\ \mathrm{s} = \boxed{13.1\ \text{hours}} \qquad [1] \qquad [3]$$

(must be in hours for the final mark)

[Due to the possible combinations of values of v and d, plus more complicated geometrical considerations we have ignored here, accept any answer that rounds to 3 s.f. to be in the range 12.9 – 13.1 hours for the full 3 marks]

**C.**

The density of the Earth: $\rho_{\oplus} = \frac{M_{\oplus}}{\frac{4}{3}\pi R_{\oplus}^{3}} = \frac{5.97\times10^{24}}{\frac{4}{3}\pi(6.37\times10^{6})^{3}} = 5514\ \mathrm{kg\ m^{-3}}$ [1]

The black hole has a radius equal to the Schwarzschild radius: $r_s = \frac{2GM}{c^2}$

$$\therefore \rho = \frac{M}{\frac{4}{3}\pi\left(\frac{2GM}{c^2}\right)^3} = \frac{3c^6}{32\pi G^3 M^2} \qquad [1]$$

$$\therefore M = \sqrt{\frac{3c^6}{32\pi G^3\rho_{\oplus}}} = \sqrt{\frac{3*(3.00\times10^{8})^{6}}{32\pi\times(6.67\times10^{-11})^{3}\times5514}}$$ (mark for rearrangement or substitution) [1]

$$= \boxed{1.15\times10^{38}\ \mathrm{kg}} \qquad (= 5.79\times10^{7}\ M_{\odot}) \qquad [1] \qquad [4]$$

**D.**

Using the cartesian convention (where u is always negative) and the thin lens formula:

$\frac{1}{u} + \frac{1}{f_o} = \frac{1}{v} \quad \therefore \frac{1}{f_o} = \frac{1}{v} - \frac{1}{u} = \frac{1}{11.2} - \frac{1}{(-28.0)} = \frac{1}{8} \quad \therefore f_o = 8 \text{ cm}$ [1]

(accept any other convention getting the same value for $f_o$)

From the properties of the telescope:

$\frac{f_o}{f_e} = x$ and $f_o + f_e = 2.5x$ [1]

$\therefore f_o + \frac{f_o}{x} = 2.5x \quad \therefore 2.5x^2 - f_o x - f_o = 0$

$\therefore 2.5x^2 - 8x - 8 = 0$ (allow ecf using their value of $f_o$) [1]

$\therefore x = 4 \ \ or \ \ x = -0.8$ Since length can't be negative, $\boxed{x = 4}$ [1] [4]

**E.**

First, we can find the redshift: $z = \frac{\Delta\lambda}{\lambda_{emit}} = \frac{683.50 - 656.28}{656.28} = 0.0415$ [1]

With this we can get the recessional velocity and hence the distance using Hubble's Law:

$$d = \frac{v}{H_0} = \frac{zc}{H_0} = \frac{0.0415 \times 3.00 \times 10^8}{70 \times 10^3} = 178 \text{ Mpc}$$ [1]

(if no further progress made you can award this mark for $zc = 1.24 \times 10^7 \text{ m s}^{-1}$ instead)

The absolute magnitude is then calculated as

$$\mathcal{M} = m - 5\log\left(\frac{d}{10}\right) = 13.84 - 5\log\left(\frac{178 \times 10^6}{10}\right) = -22.4$$ [1]

The luminosity is calculated as

$$L = 10^{-0.4(\mathcal{M} - \mathcal{M}_\odot)} L_\odot = 10^{-0.4(-22.4 - 4.74)} L_\odot = 7.24 \times 10^{10} \, L_\odot$$ [1]

Assuming all the stars are like the Sun then the mass in solar masses will be numerically the same as the luminosity in solar luminosities

$\therefore M = 7.24 \times 10^{10} \, M_\odot = \boxed{1.44 \times 10^{41} \text{ kg}}$ (accept any unit) [1] [5]

**F.**

**i)**

Since they have the same kinetic energy (and the same mass) then they have the same $v^2$, and so using the vis-viva equation:

$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 \quad \therefore v_1^2 = v_2^2 \quad \therefore GM_\odot\left(\frac{2}{r_1} - \frac{1}{a_1}\right) = GM_\odot\left(\frac{2}{r_2} - \frac{1}{a_2}\right)$ (use of vis-viva) [1]

$\therefore \frac{2}{a_1(1+e)} - \frac{1}{a_1} = \frac{2}{a_2(1-e)} - \frac{1}{a_2}$ (use of perihelion and aphelion formulae) [1]

$\therefore \frac{2a_2}{a_1(1+e)} - \frac{a_2}{a_1} = \frac{2}{1-e} - 1$

$\therefore \frac{a_2}{a_1}\left(\frac{2}{1+e} - 1\right) = \frac{2}{1-e} - 1$

$\therefore \frac{a_2}{a_1} = \frac{\frac{2}{1-e}-1}{\frac{2}{1+e}-1} = \frac{\frac{2-(1-e)}{1-e}}{\frac{2-(1+e)}{1+e}}$ (unsimplified expression for $a_2/a_1$) [1]

$= \boxed{\frac{(1+e)^2}{(1-e)^2}}$ (simplified expression) [1] [4]

**ii)**

Using the derived formula: $\frac{a_2}{a_1} = \frac{(1+e)^2}{(1-e)^2} = \frac{(1+0.5)^2}{(1-0.5)^2} = \boxed{9}$ [1] [1]

**G.**

Distance to the star: $d = \frac{1}{p} = \frac{1}{88.83\times10^{-3}} = 11.3 \text{ pc}$ [1]

Absolute magnitude: $\mathcal{M} = m - 5\log\left(\frac{d}{10}\right) = -0.05 - 5\log\left(\frac{11.3}{10}\right) = -0.307$ [1]

Luminosity:

$L = 10^{-0.4(\mathcal{M}-\mathcal{M}_\odot)}L_\odot = 10^{-0.4(-0.307-4.74)}L_\odot = 104\ L_\odot = 4.00\times10^{28}\text{ W}$ [1]

Radius of star (from geometry, using the small angle approximate and θ converted to radians):

$R = d\frac{\theta}{2} = (11.3\times3.09\times10^{16})\times\frac{\frac{21.06\times10^{-3}}{3600}\times\frac{2\pi}{360}}{2} = 1.78\times10^{10}\text{ m }(= 25.5\ R_\odot)$ [1]

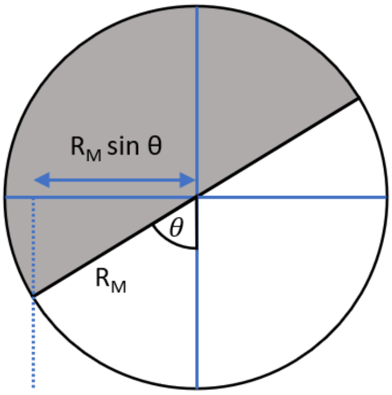
From the Stephan-Boltzmann law:

$L = 4\pi R^2\sigma T^4 \therefore T = \sqrt[4]{\frac{L}{4\pi R^2\sigma}} = \sqrt[4]{\frac{4.00\times10^{28}}{4\pi\times(1.78\times10^{10})^2\times5.67\times10^{-8}}} = 3653\text{ K}$ [1]

Finally, using Wien's displacement law:

$\lambda_{peak} = \frac{2.90\times10^{-3}}{T} = \frac{2.90\times10^{-3}}{3653} = \boxed{7.94\times10^{-7}\text{ m}}\quad (= 794\text{ nm})$ [1] [6]

**H. i)**




Suitable diagram of the situation (here it is viewed from above the lunar North pole) [1]

(OR identification of $a = R_M$ and $b = R_M \sin\theta$)

Assuming it has a circular orbit then the angular velocity will be constant:

$$\therefore \theta = \omega t = \frac{2\pi t}{T}$$ [1]

where θ is in radians, T is the lunar period and t is the time since first quarter.

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta \quad \therefore \boxed{e = \cos\left(\frac{2\pi t}{T}\right)}$$ [1] [3]

(If second mark not awarded, can get third mark for $e = \cos\theta$. Accept formulae for θ in degrees too)

**ii)**

Measuring a and b from the diagram (this will be dependent on the scale it was printed)

$$2a = 7.9 \text{ cm} \quad \text{and} \quad b = 2.8 \text{ cm} \quad \therefore e = \sqrt{1 - \frac{2.8^2}{(7.9/2)^2}} = 0.71$$ [1]

(accept measurements of *a* and *b* that give *e* in the range 0.67 to 0.74)

Time since first quarter: $t = \frac{T}{2\pi}\cos^{-1} e = \frac{29.5}{2\pi}\cos^{-1} 0.71 = 3.7 \text{ days}$ [1]

(accept 3.5-3.9 days)

Time until Full Moon: $t_{FM} = \frac{T}{4} - t = \frac{29.5}{4} - 3.7 = \boxed{3.7 \text{ days}}$ [1] [3]

(must be in days AND to 2 s.f.; accept 3.4-3.9 days)

(Due to the similarity between *t* and $t_{FM}$, look for explicit evidence of subtraction for the final mark)

**I.**

Energy needed: $\Delta Q = mc\Delta\theta = 200 \times 4.2 \times (100 - 20) = 67200 \text{ J}$ [1]

Incident intensity:

$$b_\odot = \eta \frac{L}{4\pi d^2} = 0.7 \times \frac{L_\odot}{4\pi(1 \text{ au})^2} = 0.7 \times \frac{3.83\times10^{6}}{4\pi\times(1.50\times10^{11})^2} = 948 \text{ W m}^{-2}$$ [1]

Diameter of telescope: $f\# = f_o/D$ and $m = \frac{f_o}{f_e}$ $\therefore D = \frac{mf_e}{f\#} = \frac{200\times8}{6} = 267 \text{ mm}$ [1]

Received power: $P = b_\odot \times Area = 948 \times \pi\left(\frac{0.267}{2}\right)^2 = 53.0 \text{ W}$ [1]

Time to boil: $t = \frac{\Delta Q}{P} = \frac{67200}{53.0} = 1269\ s = \boxed{21.1 \text{ min}}$ (must be in mins) [1] [5]

**J.**

From mass-energy equivalence: $\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{L}{c^2}$ [1]

Given $L \propto M^{3.5}$ $\therefore L = kM^{3.5}$ where $k = \frac{L_0}{M_0^{3.5}}$ $\therefore \frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{k}{c^2}M^{3.5} = -\frac{L_0}{M_0^{3.5}c^2}M^{3.5}$ [1]

(some students may evaluate *k* directly as $8.58 \times 10^{-80}$ W kg$^{-3.5}$)

Separating variables and integrating:

$\int_{M_0}^{\frac{1}{2}M_0} M^{-3.5}\,\mathrm{d}M = \int_0^t -\frac{L_0}{M_0^{3.5}c^2}\mathrm{d}t$ (one for separating variables, one for limits) [2]

$$\left[\frac{M^{-2.5}}{-2.5}\right]_{M_0}^{\frac{1}{2}M_0} = -\frac{L_0}{M_0^{3.5}c^2}t$$

$\therefore M_0^{-2.5}\left(-\frac{1}{2.5}\left(\frac{1}{2}\right)^{-2.5} + \frac{1}{2.5}\right) = -\frac{L_0}{M_0^{3.5}c^2}t$ (carrying out integration and inputting limits) [1]

$\therefore t = \frac{1.863 M_0 c^2}{L_0}$ (rearrangement and simplification) [1]

$\therefore t = \frac{1.863\times(7.8\times1.99\times10^{30})\times(3.00\times10^{8})^2}{3300\times3.83\times10^{26}} = \boxed{2.06\times10^{18}\ \mathrm{s}}\ (= 6.53\times10^{10}\ \mathrm{yrs})$ [1] [7]

**K.**

Finding the absolute magnitude:

$\mathcal{M} = m - 5\log\left(\frac{d}{10}\right) = 12.02 - 5\log\left(\frac{900}{10}\right) = 2.25$ [1]

Using this, we can find the effective (assuming isotropic emission) luminosity:

$L_{eff} = 10^{-0.4(\mathcal{M}-\mathcal{M}_\odot)}L_\odot = 10^{-0.4(2.25-4.74)}L_\odot = 9.92\ L_\odot$ [1]

The observed luminosity is far below this since there is only a tiny fraction of the surface area of the potential sphere of radius *d* being illuminated at any one time by the two beams:

$\therefore \frac{L_{obs}}{L_{eff}} = \frac{A_{illuminated}}{A_{total}}$ [1]

$A_{illuminated} = 2\pi d^2\left(1-\cos\frac{\theta}{2}\right)\times 2$ (for recognising the factor of 2 for 2 beams) [1]

$\therefore \frac{L_{obs}}{L_{eff}} = \frac{2\pi d^2\left(1-\cos\frac{\theta}{2}\right)\times2}{4\pi d^2} = 1-\cos\frac{\theta}{2}$ (substitution of areas) [1]

$\therefore \theta = 2\cos^{-1}\left(1-\frac{L_{obs}}{L_{eff}}\right)$ (rearrangement or substitution) [1]

$= 2\cos^{-1}\left(1-\frac{0.020}{9.92}\right) = \boxed{7.28°}$ (accept 0.127 rad) [1] [7]

Alternative route:

Observed intensity $b_{obs} = \frac{0.020\,L_\odot}{4\pi(900\ \mathrm{pc})^2} = 7.88\times10^{-16}\ \mathrm{W\ m^{-2}}$ [1]

Apparent magnitude of the Sun: $m_\odot = \mathcal{M}_\odot + 5\log(\frac{1 \text{ au in pc}}{10}) = -26.83$

Expected intensity given the observed apparent magnitude:

$b_{eff} = 10^{-0.4(m-m_\odot)} b_\odot = 10^{-0.4(12.02-(-26.63))} \times \frac{L_\odot}{4\pi(1\text{ au})^2} = 3.91 \times 10^{-13} \text{ W m}^{-2}$ [1]

Using same logic as before

$\therefore \frac{b_{obs}}{b_{eff}} = \frac{A_{illuminated}}{A_{total}}$ [1]

$A_{illuminated} = 2\pi d^2 \left(1 - \cos\frac{\theta}{2}\right) \times 2$ (for recognising the factor of 2 for 2 beams) [1]

$\therefore \frac{b_{obs}}{b_{eff}} = \frac{2\pi d^2\left(1-\cos\frac{\theta}{2}\right)\times 2}{4\pi d^2} = 1 - \cos\frac{\theta}{2}$ (substitution of areas) [1]

$\therefore \theta = 2\cos^{-1}\left(1 - \frac{b_{obs}}{b_{eff}}\right)$ (rearrangement or substitution) [1]

$= 2\cos^{-1}\left(1 - \frac{7.88\times10^{-16}}{3.91\times10^{-13}}\right) = \boxed{7.28°}$ (accept 0.127 rad) [1] [7]

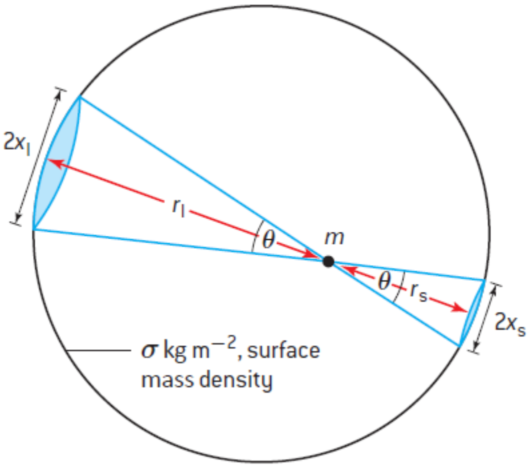
**Section 2 Mark Scheme – Q2 [Inside a Planet]**

[Throughout this question, students were told to just consider the **magnitude** of g and $V_G$, but do not penalise those that have kept (correct) negative signs in their expressions and their graphs]

a) i)

(From Newton's shell theorem) $g = \frac{GM_{enc}}{r^2}$ where $M_{enc}$ is the enclosed mass within radius $r$ and since $M_{enc} = 0$ then $\boxed{g = 0}$ (must have justification for the mark) [1] [1]

(Alternative route: use a diagram like below to show that for a test mass in a hollow sphere the attractive forces to both sides of the shell due to the mass contained within the base of the two cones are equal, so the resultant force is zero and hence $\boxed{g = 0}$)



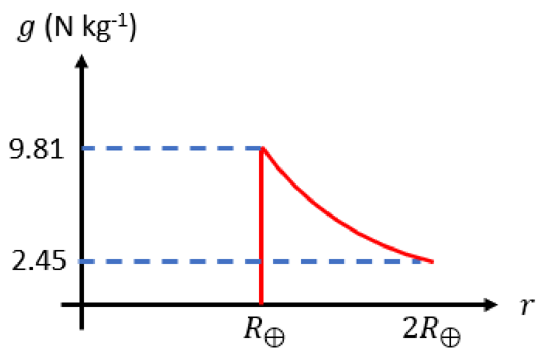

Force from large cone: $F_l = \frac{Gm(\sigma \times \pi x_l^2)}{r_l^2}$

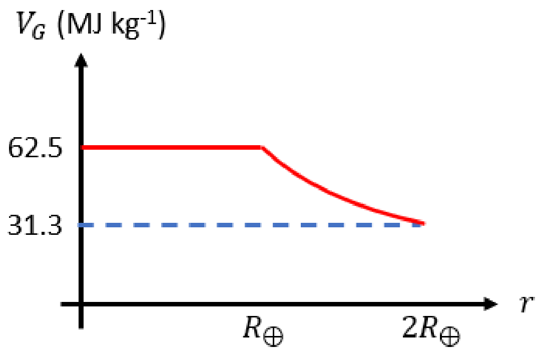
Force from small cone: $F_s = \frac{Gm(\sigma \times \pi x_s^2)}{r_s^2}$

$\tan\left(\frac{\theta}{2}\right) = \frac{x_l}{r_l} = \frac{x_s}{r_s} \quad \therefore \frac{x_l^2}{r_l^2} = \frac{x_s^2}{r_s^2} \quad \therefore F_l = F_s$

Since $g = (-)\frac{\mathrm{d}V_G}{\mathrm{d}r}$ then if g = 0 the potential $V_G$ must be a constant, equal to the value it had on the surface of the planet, so $\boxed{V_G = \frac{GM_\oplus}{R_\oplus}}$ (must have justification for the mark) [1] [1]

a) ii)




[2] [2]

(one mark for each correct graph, provided **numerical** information is given on the **y-axis**)

(allow 0.5 marks for each graph that shows the correct relationship in the region $r > R_\oplus$ i.e. $g \propto \frac{1}{r^2}$ and $V_G \propto \frac{1}{r}$)

b) i)

We can use Newton's shell theorem, taking advantage of the fact $M_{enc} = \frac{4}{3}\pi r^3 \rho$ [1]

$\therefore g = \frac{GM_{enc}}{r^2} = \frac{G\times\left(\frac{4}{3}\pi r^3 \rho\right)}{r^2} = \boxed{\frac{4}{3}\pi G \rho r}$ (accept $g = \frac{GM_\oplus}{R_\oplus^3} r$) [1] [2]

Since gravitational potential is defined as the gravitational potential energy per unit mass moving a test mass from infinity to a point in the field, then (keeping correct minus signs in for now)

$V_G = -\int_\infty^r g(r)\,\mathrm{d}r = -\int_\infty^{R_\oplus} g(r)\,\mathrm{d}r - \int_{R_\oplus}^r g(r)\,\mathrm{d}r$ (splitting into two regions) [1]

$= -\int_\infty^{R_\oplus} -\frac{GM_\oplus}{r^2}\,\mathrm{d}r - \int_{R_\oplus}^r -\frac{4}{3}\pi G \rho r\,\mathrm{d}r$

$= -\frac{GM_\oplus}{R_\oplus} + \frac{2}{3}\pi G \rho \left(r^2 - R_\oplus^2\right)$ (correct integration) [2]

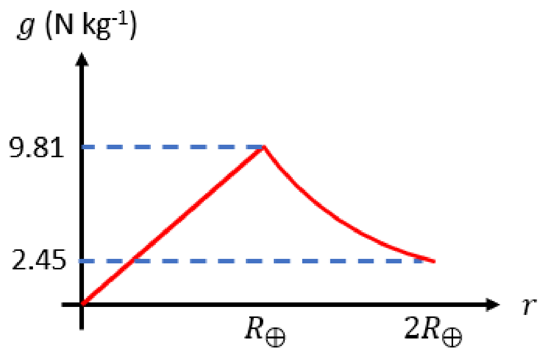
But since the density is constant, $\rho = \frac{M_\oplus}{\frac{4}{3}\pi R_\oplus^3}$

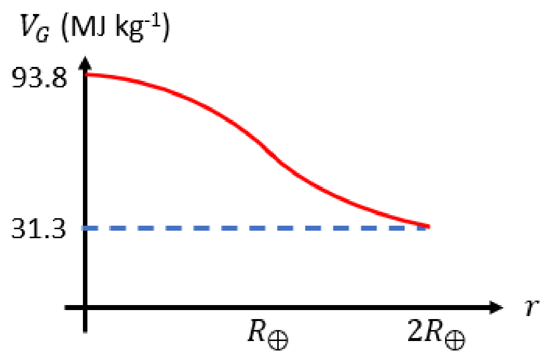
$\therefore V_G = -\frac{GM_\oplus}{R_\oplus} + \frac{2}{3}\pi G \frac{M_\oplus}{\frac{4}{3}\pi R_\oplus^3} r^2 - \frac{2}{3}\pi G \frac{M_\oplus}{\frac{4}{3}\pi R_\oplus^3} R_\oplus^2 = -\frac{3GM_\oplus}{2R_\oplus} + \frac{GM_\oplus}{2R_\oplus^3} r^2$

$\therefore$ the magnitude of $V_G$ is $\boxed{V_G = \frac{3GM_\oplus}{2R_\oplus} - \frac{GM_\oplus}{2R_\oplus^3} r^2}$ (do not penalise if not magnitude) [1] [4]

(If left in terms of ρ, only get max 3 marks UNLESS the y-intercept in $V_G$ graph has the correct numerical value, in which case they can get 4 marks. If they only integrate their expression for g for $r < R_\oplus$ then max 2 marks)

b) ii)




[2] [2]

(one mark for each correct graph, provided **numerical** information is given on the **y-axis**)

(if no / incorrect numerical information on y-axis allow 0.5 marks for linear section with positive gradient of $g - r$ graph and 0.5 marks for parabolic section of $V_G - r$ graph [whether inverted or not])

c) i)

Our expression for the density as a function of r is $\rho = \rho_c\left(1 - \frac{r}{R_\oplus}\right)$ [1]

Finding the enclosed mass as a function of radius by considering a sequence of concentric thin shells of thickness dr

$$M_{enc} = \int_0^r \rho_c\left(1 - \frac{r}{R_\oplus}\right) 4\pi r^2\,\mathrm{d}r = 4\pi\rho_c \int_0^r r^2 - \frac{1}{R_\oplus}r^3\,\mathrm{d}r = 4\pi\rho_c\left(\frac{1}{3}r^3 - \frac{1}{4R_\oplus}r^4\right) \quad [1]$$

$$\therefore g = \frac{GM_{enc}}{r^2} = \frac{G\times\left(4\pi\rho_c\left(\frac{1}{3}r^3 - \frac{1}{4R_\oplus}r^4\right)\right)}{r^2} = \boxed{4\pi G\rho_c\left(\frac{1}{3}r - \frac{1}{4R_\oplus}r^2\right)} \quad [1] \quad [3]$$

c) ii)

We can find the maximum value through differentiation:

$$\frac{\mathrm{d}g}{\mathrm{d}r} = \frac{4}{3}\pi G\rho_c\left(\frac{1}{3} - \frac{1}{2R_\oplus}r\right) \quad [1]$$

For the turning point, we set the differential equal to zero:

$$\therefore \frac{1}{3} - \frac{1}{2R_\oplus}r = 0 \quad \therefore \boxed{r = \frac{2}{3}R_\oplus} \quad [1] \quad [2]$$

Plugging into our expression for $g$:

$$g_{max} = 4\pi G\rho_c\left(\frac{1}{3}\left(\frac{2}{3}R_\oplus\right) - \frac{1}{4R_\oplus}\left(\frac{2}{3}R_\oplus\right)^2\right) = \boxed{\frac{4}{9}\pi G\rho_c R_\oplus} \quad [1] \quad [1]$$

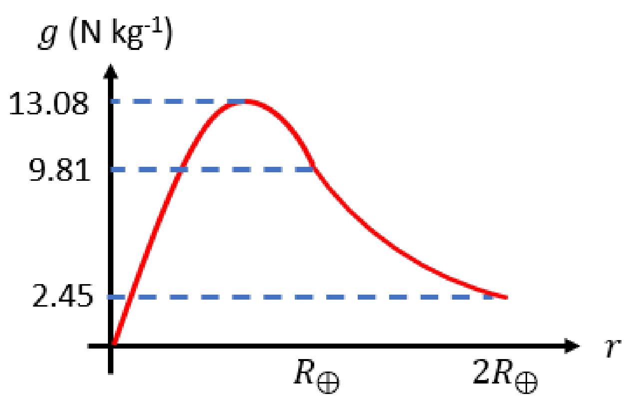
c) iii)

Since $\rho_c$ is unknown, we need to find a way to evaluate $g_{max}$ – one strategy is to use the known value of $g$ at the surface, $g_0 = 9.81\ \mathrm{N\ kg^{-1}}$

$$g_0 = g(R_\oplus) = 4\pi G\rho_c\left(\frac{1}{3}R_\oplus - \frac{1}{4R_\oplus}R_\oplus^2\right) = \frac{1}{3}\pi G\rho_c R_\oplus$$

$$\therefore g_{max} = \frac{4}{3}g_0 = \frac{4}{3}\times 9.81 = 13.08\ \mathrm{N\ kg^{-1}} \quad [1]$$

(Alternative: use $g_0 = \frac{1}{3}\pi G\rho_c R_\oplus$ to calculate $\rho_c = 2.20\times 10^4\ \mathrm{kg\ m^{-3}}$ and hence calculate $g_{max}$)




(This mark is for a correctly shaped graph with **numerical** information on the **y-axis**) [1] [2]

## Section 2 Mark Scheme – Q3 [Jeans Mass]

a)

Considering an infinitely thin shell of radius $r$ and thickness $\mathrm{d}r$

$GPE_{shell} = -\frac{Gm_{shell}m_{enclosed}}{r}$ [1]

But $m_{shell} = 4\pi r^2 \rho \, \mathrm{d}r$ [1]

$\therefore GPE_{shell} = -\frac{G}{r}(4\pi r^2 \rho \, \mathrm{d}r)\left(\frac{4}{3}\pi r^3 \rho\right) = -\frac{16\pi^2\rho^2 G r^4}{3}\,\mathrm{d}r$ [1]

$\therefore GPE_{tot} = \int_0^R -\frac{16\pi^2\rho^2 G r^4}{3}\,\mathrm{d}r = \left[-\frac{16\pi^2\rho^2 G r^5}{15}\right]_0^R = -\frac{16\pi^2\rho^2 G R^5}{15}$ [2]

(Mark for limits, mark for correct integration)

Since $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ we can substitute it in:

$GPE_{tot} = -\frac{16\pi^2 G R^5}{15}\left(\frac{3M}{4\pi R^3}\right)^2 = -\frac{3GM^2}{5R}$ [1]

so $\boxed{\alpha = \frac{3}{5}}$ (Watch out that alpha is **positive**) [1] [7]

[An alternative is to sub in the expression for $\rho$ earlier and hence have $GPE_{shell} = -\frac{3GM^2r^4}{R^6}\,\mathrm{d}r$. This route also gains full credit.]

b)

$KE_{tot} = \frac{3}{2}Nk_BT = \frac{3}{2}\frac{M_J}{m_H}k_BT$ (Mark for KE expression, mark for removing N) [2]

Plugging this into the virial theorem:

$\frac{3M_Jk_BT}{2m_H} = -\frac{1}{2}\left(-\frac{3GM^2}{5R}\right) = \frac{3GM_J^2}{10R}$ [1]

$\therefore M_J = \frac{5Rk_BT}{Gm_H}$ [1]

But since $M_J = \frac{4}{3}\pi R^3 \rho \Rightarrow R = \left(\frac{3M_J}{4\pi\rho}\right)^{1/3}$ [1]

Substituting in: $M_J = \frac{5k_BT}{Gm_H}\left(\frac{3M_J}{4\pi\rho}\right)^{1/3}$ [1]

$\therefore \boxed{M_J = \left(\frac{5k_BT}{Gm_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}}$ (Must be only in terms of T, ρ and constants) [1] [7]

[Accept other algebraically equivalent forms, e.g. $M_J = \left(\frac{375k_B^3T^3}{4\pi\rho G^3 m_H^3}\right)^{1/2}$ for full credit.]

If using $\alpha = 1$ then $M_J = \frac{3Rk_BT}{Gm_H} = \boxed{\left(\frac{3k_BT}{Gm_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}}$.

c)

Reading off the graph: $\lambda_{peak} = 0.105\ \text{cm}$ (accept 0.100 cm to 0.110 cm) [1]

From Wien's displacement law, $T_{CMB} = \frac{2.90\times10^{-3}}{0.105\times10^{-2}} = 2.76\ \text{K}$ [1]

Using the given density information, $\rho = \frac{m_H}{4} = \frac{1.67\times10^{-27}}{4} = 4.18\times10^{-28}\ \text{kg m}^{-3}$ [1]

$$\therefore M_J = \left(\frac{5k_BT}{Gm_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{\frac{1}{2}}$$

$$= \left(\frac{5\times1.38\times10^{-23}\times2.76}{6.67\times10^{-11}\times1.67\times10^{-27}}\right)^{3/2}\left(\frac{3}{4\pi\times4.18\times10^{-28}}\right)^{1/2} = 1.69\times10^{36}\ \text{kg}$$ [1]

$$\therefore R = \left(\frac{3M_J}{4\pi\rho}\right)^{1/3} = \left(\frac{3\times1.69\times10^{36}}{4\pi\times4.18\times10^{-28}}\right)^{1/3} = 9.89\times10^{20}\ m$$ [1]

$$= \boxed{32.0\ \text{kpc}}$$ (must be in kpc for final mark) [1] [6]

[An alternative is to sub in the expression for $M_J$ into the one for R to get $R = \left(\frac{15k_BT}{4\pi\rho Gm_H}\right)^{1/2}$]

If using $\alpha = 1$ then $M_J = 7.86\times10^{35}\ \text{kg}$ and $R = \left(\frac{9k_BT}{4\pi\rho Gm_H}\right)^{1/2} = 7.66\times10^{20}\ \text{m} = \boxed{24.8\ \text{kpc}}$

**Section 2 Mark Scheme – Q4 [Spherical Telescope]**

a)

The Cartesian equation of the ray reflected at the origin: $y = -\frac{1}{\tan\theta}x$ [1]

The Cartesian equation of the other reflected ray:

$y - r(1-\cos\phi) = \frac{1}{\tan(2\phi-\theta)}(x + r\sin\phi)$ (this is in the form $y - y_0 = m(x - x_0)$) [1]

(Accept algebraically equivalent expressions)

At the intersection we can use the first equation to eliminate $x$ in the second equation:

$y - r(1-\cos\phi) = \frac{1}{\tan(2\phi-\theta)}(-y\tan\theta + r\sin\phi)$ [1]

Rearranging to get $y$ on the LHS:

$y\left(1 + \frac{\tan\theta}{\tan(2\phi-\theta)}\right) = \frac{r\sin\phi}{\tan(2\phi-\theta)} + r(1-\cos\phi)$ (gather y terms on LHS) [1]

$y = r\frac{\sin\phi + \tan(2\phi-\theta)(1-\cos\phi)}{\tan(2\phi-\theta)+\tan\theta}$ (simplify) [1]

Using the given tangent addition of angles formula (with $\tan(-\theta) = -\tan(\theta)$)

$y = r\frac{\sin\phi + \frac{\tan 2\phi - \tan\theta}{1+\tan 2\phi\tan\theta}(1-\cos\phi)}{\frac{\tan 2\phi-\tan\theta}{1+\tan 2\phi\tan\theta}+\tan\theta} = r\frac{\sin\phi\,(1+\tan 2\phi\tan\theta) + (\tan 2\phi - \tan\theta)(1-\cos\phi)}{\tan 2\phi - \tan\theta + \tan\theta\,(1+\tan 2\phi\tan\theta)}$ [1]

We are told that $r$ is much larger than the dimensions of the mirror, so $\phi \ll 1$ rad

Hence, using the small angle approximations that $\sin\phi \approx \phi$, $\cos\phi \approx 1$ and $\tan 2\phi \approx 2\phi$ and ignoring any terms of order $\phi^2$ and higher

$y = r\frac{\phi(1+2\phi\tan\theta)+(2\phi-\tan\theta)(1-1)}{2\phi+2\phi\tan^2\theta} = r\frac{\phi+2\phi^2\tan\theta}{2\phi(1+\tan^2\theta)} \approx r\frac{\phi}{2\phi(1+\tan^2\theta)} = \boxed{r\frac{1}{2(1+\tan^2\theta)}}$ [3] [9]

This can be simplified further using one last trigonometric identity (but is not required)

$y = \frac{r}{2}\frac{1}{1+\tan^2\theta} = \frac{r}{2}\frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \frac{r}{2}\frac{\cos^2\theta}{\cos^2\theta+\sin^2\theta} = \boxed{\frac{r}{2}\cos^2\theta}$

(could make use of the identity $1+\tan^2\theta = \sec^2\theta$ as an alternative to showing it explicitly)

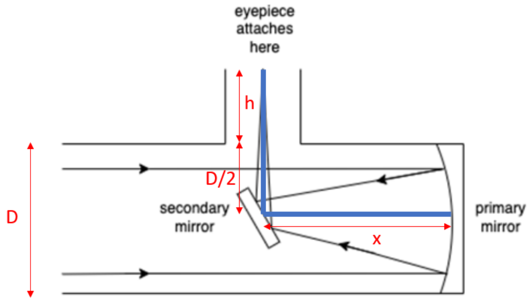
Breakdown:
- MP1 & 2 – Equations of reflected rays
- MP3, 4 & 5 – Intersection of lines, fully simplified into the form $y = f(r,\theta,\phi)$
- MP6 – Use of given tangent addition of angles formula
- MP7 – Use of small angle approximation
- MP8 – Ignoring of terms second order and higher
- MP9 – Final answer, fully simplified, in form $y = f(r,\theta)$

b)

In the paraxial approximation $\theta \approx 0$ $\therefore y = \frac{r}{2}$ but since $y \to f$ as $\theta \to 0$ $\therefore \boxed{f = \frac{r}{2}}$ [1] [1]

c)

Appropriate ray diagram showing how the focal length links to other distances when the mirror is taken into account [1]




e.g.

The focal length is the length of the blue line, which is $f = x + \frac{D}{2} + h$ [1]

Statement (or use of the idea) that since the mirror is at 45°, the angle between the ray and the (rotated) principal axis is unaltered and so the focal length is unaffected by its presence [1]

$\therefore x = f - \frac{D}{2} - h = 150 - \frac{20}{2} - 5 = \boxed{135\text{ cm}}$ (could be given in metres instead) [1] [4]

d) i)

$f\# = \frac{f}{D} = \frac{150}{20} = \boxed{7.5}$ [1] [1]

d) ii)

Using the given formula for the Rayleigh criterion:

$\theta = \frac{1.22\lambda}{D} = \frac{1.22\times600\times10^{-9}}{0.2} = 3.66\times10^{-6}\text{ rad}$ [1]

$= \frac{3.66\times10^{-6}}{2\pi}\times360\times3600 = \boxed{0.755''}$ (accept 1 s.f.) [1] [2]

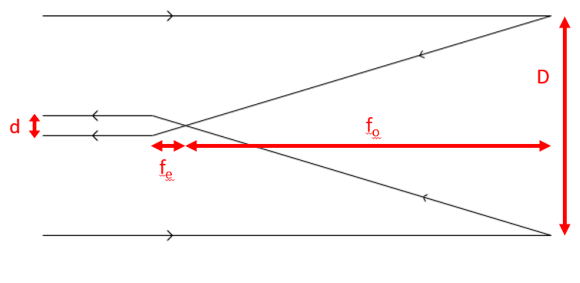
d) iii)

The (linear) magnification of the telescope: $M = \frac{f_o}{f_e} = \frac{150\text{ cm}}{1\text{ cm}} = 150$ [1]

(Use of the) idea that the light intensity is inversely proportional to the area of the beam [1]

Areal magnification = (linear magnification)$^2$ $\therefore$ the factor increase is $150^2 = \boxed{22500}$ [1] [3]

Alternative route:




Comparing brightness: $\frac{b_{new}}{b_{old}} = \frac{L/A_{new}}{L/A_{old}} = \frac{A_{old}}{A_{new}}$

By considering areas, $\frac{A_{old}}{A_{new}} = \frac{\pi\left(\frac{D}{2}\right)^2}{\pi\left(\frac{d}{2}\right)^2} = \frac{D^2}{d^2}$

Due to similar triangles, $\frac{D^2}{d^2} = \frac{f_o^2}{f_e^2} = \frac{150^2}{1^2} = \boxed{22500}$

## Section 2 Mark Scheme – Q5 [Dark Energy]

a)

Converting $H_0$ into SI base units: $\frac{70\times10^3}{10^6\times3.09\times10^{16}} = 2.27\times10^{-18}\ \mathrm{s^{-1}}$ [1]

The value of the cosmological constant:

$\Lambda = 3\left(\frac{H_0}{c}\right)^2\Omega_\Lambda = 3\left(\frac{2.27\times10^{-18}}{3.00\times10^8}\right)^2\times0.7 = \boxed{1.20\times10^{-52}}$ [1] [2]

SI base units: $\boxed{\mathrm{m^{-2}}}$ [1] [1]

b)

Since travelling at escape velocity

$\therefore -\frac{GMm}{r}+\frac{1}{2}mv^2 = 0 \quad \therefore \frac{GM}{r} = \frac{1}{2}v^2$ (mark for idea that $E_{tot} = 0$) [1]

Substituting in $v = H_0 r$ and $M = \frac{4}{3}\pi r^3\rho_{crit}$

$\frac{4\pi G r^2\rho_{crit}}{3} = \frac{1}{2}H_0^2 r^2$ (mark for introducing $H_0$, mark for introducing ρ) [2]

Cancel the $r^2$ and rearrange

$\rho_{crit} = \frac{3H_0^2}{8\pi G} \quad \therefore \boxed{\sigma = \frac{3}{8\pi}}$ [1] [4]

c)

Evaluating $\rho_{crit}$: $\rho_{crit} = \frac{3\times(2.27\times10^{-18})^2}{8\pi\times6.67\times10^{-11}} = 9.18\times10^{-27}\ \mathrm{kg\ m^{-3}}$ [1]

Given $m_H \approx m_p = 1.67\times10^{-27}\ \mathrm{kg}$ $\therefore \boxed{\rho_{crit} = 5.50\ \mathrm{H\ atoms\ m^{-3}}}$ [1] [2]

d)

Given $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{crit}}$ $\therefore \rho_\Lambda = \Omega_\Lambda\rho_{crit} = 0.7\times9.18\times10^{-27} = \boxed{6.43\times10^{-27}\ \mathrm{kg\ m^{-3}}}$ [1] [1]

e)

Expressing all quantities in SI base units [1]

$[E] = \mathrm{J} = \mathrm{kg\,m^2\,s^{-2}}$ $[\hbar] = \mathrm{J\,s} = \mathrm{kg\,m^2\,s^{-1}}$ $[c] = \mathrm{m\,s^{-1}}$ $[G] = \mathrm{N\,m^2\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$

Setting up the dimensional analysis equations by comparing powers of units on the LHS and RHS

$$\mathrm{kg\,m^2\,s^{-2}} = (\mathrm{kg\,m^2\,s^{-1}})^{\alpha} \times (\mathrm{m\,s^{-1}})^{\beta} \times (\mathrm{kg^{-1}\,m^3\,s^{-2}})^{\gamma}$$

kg: $1 = \alpha - \gamma$ m: $2 = 2\alpha + \beta + 3\gamma$ s: $-2 = -\alpha - \beta - 2\gamma$ [1]

These equations solve to give: $\boxed{\alpha = \frac{1}{2},\ \beta = \frac{5}{2},\ \gamma = -\frac{1}{2}}$ [1] [3]

$$\therefore E_{Pl} = \hbar^{\frac{1}{2}} c^{\frac{5}{2}} \gamma^{-\frac{1}{2}} = \left(\frac{6.63\times10^{-34}}{2\pi}\right)^{\frac{1}{2}} \times (3.00 \times 10^{8})^{\frac{5}{2}} \times (6.67 \times 10^{-11})^{-\frac{1}{2}}$$

$$= \boxed{1.96 \times 10^{9}\ \mathrm{J}}$$ (must be in joules) [1] [1]

f)

$$\omega_{max} = \frac{E_{Pl}}{\hbar} = \frac{1.96\times10^{9}}{\left(\frac{6.63\times10^{-34}}{2\pi}\right)} = \boxed{1.86 \times 10^{43}\ \mathrm{rad\,s^{-1}}}$$ [1] [1]

(must have unit; accept $\mathrm{s^{-1}}$ and Hz)

$$u_{vac} = \frac{\hbar}{8\pi^2 c^3}\omega_{max}^4 = \frac{\left(\frac{6.63\times^{-34}}{2\pi}\right)}{8\pi^2\times(3.00\times10^8)^3} \times (1.86 \times 10^{43})^4 = \boxed{5.90 \times 10^{111}}$$ [2] [2]

(Due to its size, most calculators will not be able to evaluate $u_{vac}$ directly so the first mark of the two can be given for showing a suitable method

e.g. $\log_{10} u_{vac} = \log_{10}\left(\frac{\hbar}{8\pi^2 c^3}\right) + 4\log_{10}\omega_{max} = 111.77\ldots$ etc.)

Units: $\boxed{\mathrm{J\,m^{-3}}}$ [1] [1]

g)

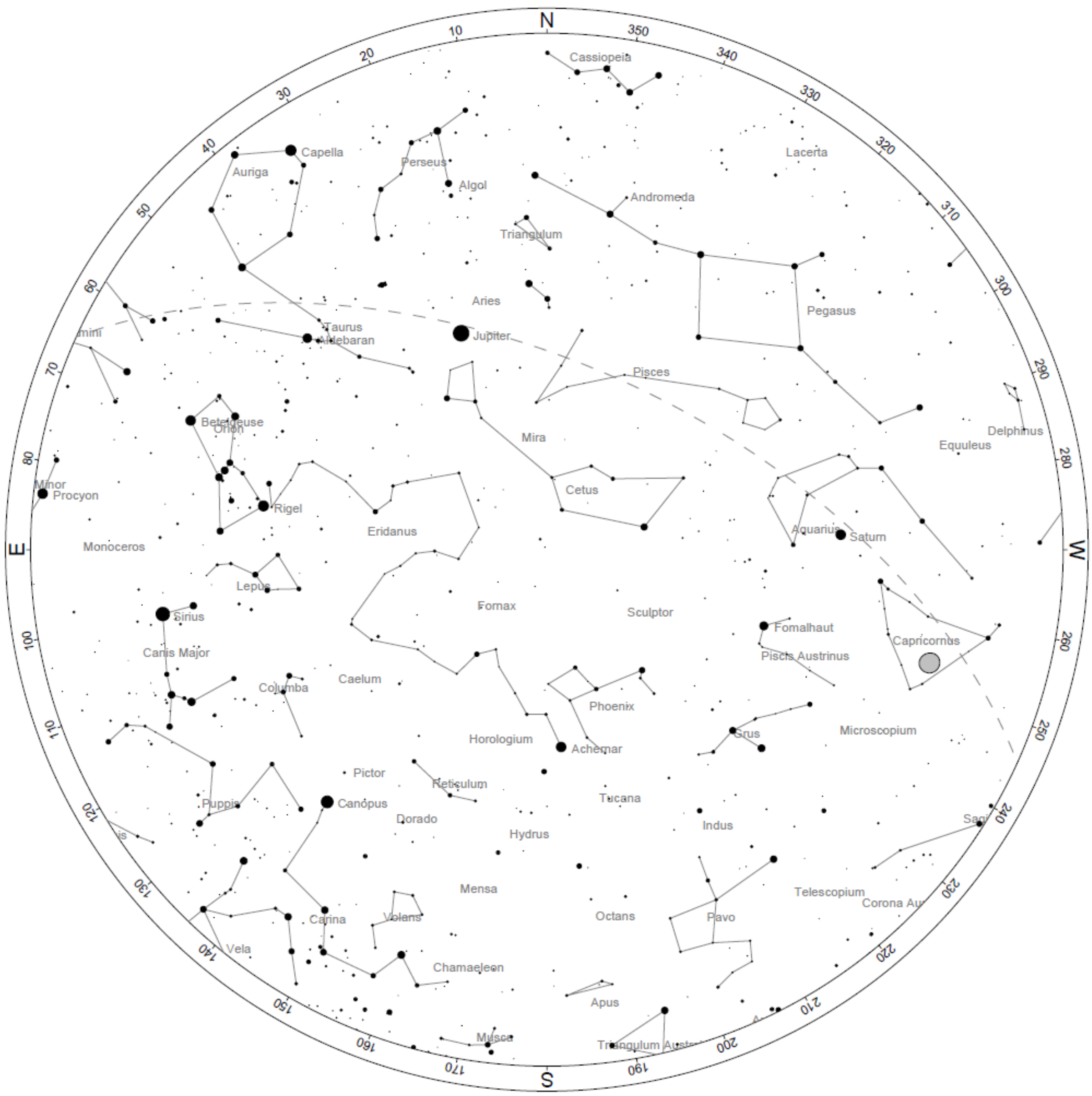
$$\rho_{vac} = \frac{u_{vac}}{c^2} = \frac{5.90\times10^{111}}{(3.00\times10^8)^2} = \boxed{6.56 \times 10^{94}\ \mathrm{kg\,m^{-3}}}$$ [1] [1]

h)

$$\log_{10}\left(\frac{\rho_{vac}}{\rho_{\Lambda}}\right) = \log_{10}\left(\frac{6.56\times10^{94}}{6.43\times10^{-27}}\right) = \log_{10}(1.02 \times 10^{121}) = \boxed{121.01\ldots}\quad (\approx 120)$$ [1] [1]

(Since it is a ‘show that’ expect to see **at least** 3 s.f.)

## Section 2 Mark Scheme – Q6 [Observational Astronomy]




a) A – Alpheratz (accept α And) [1] [1]

B – Rigel (accept β Ori) [1] [1]

C – Aldebaran (accept α Tau) [1] [1]

b) Altitude = 22° (accept 20° to 24°) [1] [1]

Nearest marker = 100° [1] [1]

c) Expect:

Perseus, Orion, Canis Major [NOT Taurus, as wanted non-zodiacal] [3] [3]

(Accept Cassiopeia and Auriga too since their main body is above the horizon, even if their full boundary is not. For others, verify against map on next page with galactic equator in blue)

d) Taurus, Aries [2] [2]

e) Capricornus [1] [1]

f) i) 0.5 marks for each correct answer

| | | | |
|---|---|---|---|
| M1 – ABOVE | (in Taurus) | M45 – ABOVE | (in Taurus) |
| M31 – ABOVE | (in Andromeda) | M57 – BELOW | (in Lyra) |
| M42 – ABOVE | (in Orion) | M76 – ABOVE | (in Perseus) [3] [3] |

ii) Andromeda Galaxy, M31 [2] [2]

g) [This is just one of several methods you could use]

Aries is on the meridian [1]

At midnight 3 hours ago, Aquarius will have been on the meridian [1]

So, the Sun is directly opposite Aquarius, in Leo [1]

This means the month is $\boxed{\text{August}}$ [1] [4]

[This is the sky above Rio de Janeiro, Brazil, on 29th August 2023. Numerical methods using RA that get early September will also be given full credit]